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A Geometric Problem in Approximation Theory

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1.

Let S be a compact subset of Euclidean n-space E_n. Let $C(S)$ be the space of real-valued continuous functions on S and $\mathscr{P}_k$ the finite dimensional subspace of real-valued polynomials of degree $\leqslant k$, $0 \leqslant k < \infty$. For $f, g \in C(S)$, $d(f, g) = \|f - g\| = \operatorname{Sup}_{x \in S} |f(x) - g(x)|$ is called the sup-norm distance between f and g. The distance $d(f, \mathscr{P}_k)$ between f and $\mathscr{P}_k$ is defined to be $\operatorname{Min}_{p \in \mathscr{P}_k} d(f, p)$. $d(f, \mathscr{P}_k)$ is attained on $\mathscr{P}_k$, i.e., $\exists p_k \in \mathscr{P}_k \ni d(f, p_k) = d(f, \mathscr{P}_k)$, any such p_k being called a best kth degree approximant to f on S [1, p. 20]. In general, p_k is not unique (see, in this respect, Haar's Unicity Theorem [1, p. 81]).

We ask the following question. Suppose $f(x)$ is independent of some variable x_i. Can p_k, for all k, also be chosen to be independent of x_i? We shall characterize the sets for which this is the case.

For $n = 1$ the problem is trivial and we let $n \geqslant 2$. We find it convenient to replace n by $n + 1$, so that $n \geqslant 1$, and to denote the points of E_{n+1} by $\tilde{x} = (x, y)$, where $x = (x_1, \ldots, x_n)$. Without loss of generality, we assume the above-mentioned variable x_i to be y. $\tilde{S}$ is used to denote any compact subset of $E_{n \pm 1}$, and S to denote the set $\{x : \tilde{x} \in \tilde{S}\}$. For $a \in S$, the set $\tilde{S}_a = \{\tilde{x} \in S : x = a\}$ will be referred to as the vertical section of $\tilde{S}$ based at a. S is called the projection of $\tilde{S}$. Observe that $f(x) \in C(S)$ iff $f(x) \in C(\tilde{S})$.

DEFINITION 1. (i) $\tilde{S}$ has property H, or $S \in H$, iff $\forall f(x) \in C(S)$ and $\forall k$, $0 \leqslant k < \infty$, $\exists$ a best kth-degree approximant $p_k(x)$ to $f(x)$ on $C(\tilde{S})$. That is, if f is independent of y, then, for all k, p_k may be chosen to be independent of y. $p_k(x)$ is thus a best kth-degree approximant to $f(x)$ on both S and $\tilde{S}$.

(ii) $\tilde{S}$ has property H_k, or $\tilde{S} \in H_k$, $0 \leqslant k < \infty$, iff $\forall f(x) \in C(S)$ and $\forall j$, $0 \leqslant j \leqslant k$, $\exists$ a best jth-degree approximant $p_k(x)$ to $f(x)$ on $\tilde{S}$. Namely, if f is independent of y, then p_j may be chosen to be independent of y, provided $j \leqslant k$.

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Clearly H_0 consists of all compact subsets of E_{n+1} and $H_k \downarrow H = \bigcap_{k=0}^{\infty} H_k$. We use some results in approximation theory to translate the property H_k into algebraic terms (Theorem 2). For $k = 1$, Helly's Theorem on convex sets is then employed to convert the algebraic criterion into the following geometric one.

THEOREM 5. *$\tilde{S} \in H_1$ iff $\exists$ a hyperplane $y = a_1 x_1 + \cdots + a_n x_n + a_{n+1}$ which meets the convex hulls of all vertical sections of $\tilde{S}$.*

In Theorem 6, we show that the geometric characterization of H_1 is also one of H_k, $k \geqslant 2$, provided $\tilde{S}$ meets every vertical line $x = \text{constant}$ in an interval, in which case $\tilde{S}$ is called y-convex. In Theorem 7, some examples are given showing that Theorem 6 need not hold when $\tilde{S}$ is not y-convex, and we have no geometric characterization of H_k, $k \geqslant 2$, valid for all compact sets.

Finally, we mention that the problem which we have posed can be generalized to the case where x is obtained from $\tilde{x}$ by deleting more than one variable. Our methods are not amenable to this case, which we leave as an open problem. The difficulty with the generalized problem is that there seems to be no appropriate counterpart to Theorem 3, which is crucial to our argument.

2.

We introduce several concepts which prove useful in characterizing H_k.

DEFINITION 2. (i) Let $\mathscr{h}(\tilde{S}_a) =$ convex hull of $\tilde{S}_a$, $\mathscr{h}_y(\tilde{S}) = \bigcup_{a \in S} \mathscr{h}(\tilde{S}_a)$. $\mathscr{h}_y(\tilde{S})$ is called the y-convex hull of $\tilde{S}$.

(ii) $\tilde{S}$ is y-convex iff $\mathscr{h}_y(\tilde{S}) = \tilde{S}$. That is, $\tilde{S}$ is y-convex iff every vertical line meets it in an interval.

The term hyperplane of E_{n+1} denotes any subset π of E_{n+1} with equation $a_1 x_1 + \cdots + a_n x_n + a_{n+1} y + a_{n+2} = 0$, the a_k constants not all zero. π is called a nonvertical hyperplane or n.v.h. if its projection is E_n, i.e., $a_{n+1} \neq 0$.

DEFINITION 3. $\tilde{S}$ has the transversal property (t.p) iff $\exists$ n.v.h. meeting all its vertical sections.

DEFINITION 4. Let Σ be a finite subset of E_n and $\tilde{\Sigma}$ a subset of E_{n+1} projecting onto Σ. Let $\varepsilon(x)$ be a function on Σ with values ± 1, and $\tilde{\varepsilon}(\tilde{x}) = \varepsilon(x)$, $\tilde{x} \in \tilde{\Sigma}$. $\varepsilon(x)$ is called a signature in E_n with carrier Σ, and $\tilde{\varepsilon}(\tilde{x})$ a lifted signature in E_{n+1}, or the lifting of $\varepsilon(x)$ to E_{n+1}, with carrier $\tilde{\Sigma}$.

(ii) A signature $\varepsilon(x)$ is admissible of degree k iff $\exists$ a polynomial $p_k(x)$ of degree $\leqslant k \ni \varepsilon(x)\, p_k(x) > 0,\ x \in \Sigma$. A lifted signature $\tilde{\varepsilon}$ is admissible of degree k iff $\exists$ a polynomial $p_k(\tilde{x})$ of degree $\leqslant k \ni \tilde{\varepsilon}(\tilde{x})\, p_k(\tilde{x}) > 0,\ \tilde{x} \in \tilde{\Sigma}$. ε and $\tilde{\varepsilon}$ are called inadmissible of degree k if the above requirements do not hold.

Remarks. Theorem 1 below brings out the significance of inadmissible signature for approximation theory. The geometric characterization of inadmissible signatures and lifted signatures is a difficult problem. Two noteworthy exceptions are:

(i) $k=1$: Let $\Sigma_+ = \{x\colon \varepsilon(x) = +1\}$, $\Sigma_- = \{x\colon \varepsilon(x) = -1\}$. $\varepsilon(x)$ is inadmissible of degree 1 iff there is no hyperplane in E_n separating Σ_+ from Σ_-, which is equivalent to $\mathscr{h}(\Sigma_+) \cap \mathscr{h}(\Sigma_-) \neq \varnothing$ [2, p. 21, exercise 1]. A similar description holds for $\tilde{\varepsilon}$.

(ii) $n=1$: $\varepsilon(x)$ is inadmissible of degree k iff Σ contains $k+2$ points $x_1 < \cdots < x_{k+2}$ with either $\varepsilon(x_i) = (-1)^i\ \forall i$ or $\varepsilon(x_i) = (-1)^{i+1}\ \forall i$ [5, p. 682].

THEOREM 1 [5, p. 678]. *Let $f(x) \in C(S)$, $f(x) \neq$ polynomial, and $p_k(x)$ a polynomial of degree $\leqslant k$. $p_k(x)$ is a best kth-degree approximant to $f(x)$ on S iff $\exists$ inadmissible signature $\varepsilon(x)$ of degree k with carrier $\Sigma \subset S \ni$ $\varepsilon(x)(f(x) - p_k(x)) = \|f(x) - p_k(x)\|$, $x \in \Sigma$.*

We obtain a criterion for $\tilde{S}$ to be in H_k. We say that $\varepsilon(x)$ is a signature in S if $\Sigma \subset S$. In this case $\tilde{\Sigma}$ is assumed to be $\{\tilde{x} \in \tilde{S}\colon x \in \Sigma\}$. $\tilde{\varepsilon}(\tilde{x})$ is then called a lifted signature in $\tilde{S}$.

THEOREM 2. *Let $1 \leqslant k < \infty$. $\tilde{S} \in H_k$ iff every inadmissible signature of degree k in S lifts to an inadmissible signature of degree k in $\tilde{S}$.*

Proof. We prove the equivalence of the negatives of the two statements. Suppose that $\tilde{S} \notin H_k$. Namely, $\exists f(x) \in C(S)$, with $p_k(x)$ as a best kth-degree approximant on S, and a polynomial $q_k(\tilde{x})$ of degree $\leqslant k \ni$

$$\|f(x) - p_k(x) - q_k(\tilde{x})\| < \|f(x) - p_k(x)\|. \tag{2.1}$$

Choose $\varepsilon(x)$ as in Theorem 1. It follows from (2.1) that $f(x) - p_k(x)$ and $q_k(\tilde{x})$ have identical signs on $\tilde{\Sigma}$. This means that $\tilde{\varepsilon}(\tilde{x})\, q_k(\tilde{x}) > 0$, $\tilde{x} \in \tilde{\Sigma}$, so that $\tilde{\varepsilon}(\tilde{x})$ is admissible of degree k in $\tilde{S}$.

Suppose that $\varepsilon(x)$ is an inadmissible signature of degree k in S which lifts to an admissible signature $\tilde{\varepsilon}(\tilde{x})$ of degree k in $\tilde{S}$. Let $f(x) = \varepsilon(x)$ on Σ and extend $f(x)$ to be continuous on S with $|f(x)| < 1$, $x \in S - \Sigma$. By Theorem 1 0 is a best kth-degree approximant to $f(x)$ on S. Let $q_k(\tilde{x})$ be a polynomial of degree $\leqslant k \ni \tilde{\varepsilon}(\tilde{x})\, q_k(\tilde{x}) > 0$, $\tilde{x} \in \tilde{\Sigma}$. For $\delta > 0$ sufficiently small, $\|f(x) - \delta q_k(\tilde{x})\| < \|f(x) - 0\|$. Hence, $S \notin H_k$.

In view of Remark (ii), Theorem 2 can be restated as follows when $n = 1$.

THEOREM 2′. *Let* $n = 1$, $1 \leqslant k < \infty$. $\tilde{S} \in H_k$ *iff for every set of points* $x_1 < \cdots < x_{k+2}$ *in* S, *there is no polynomial* $p_k(x, y)$ *of degree* $\leqslant k$ *satisfying* $(-1)^i p_k(x, y) > 0$ *along* S_{x_i}, $1 \leqslant i \leqslant k+2$.

For $k > 1$, the criterion of Theorem 2 is ineffective, as it is difficult to decide whether a given signature or lifted signature is inadmissible of degree k. For $k = 1$, however, we have the geometric description given in Remark (i). We use it to obtain a geometric description of H_1. We require Helly's Theorem and its consequences Theorems 3 and 4.

HELLY'S THEOREM [4]. *Let* $\{C_\alpha\}$, $\alpha \in A$ *and* $|A| \geqslant n+1$, *be a family of closed convex subsets of* E_n. *Suppose that*

(i) $\exists$ *a finite number of* C_α*'s with nonempty bounded intersection.*

(ii) *Any* $n+1$ C_α*'s have nonempty intersection.*

Then $\bigcap_{\alpha \in A} C_\alpha \neq \varnothing$. *If* A *is finite then the theorem holds if the* C_α*'s are just assumed convex.*

We use the following terminology. An m-flat in E_n, $0 \leqslant m \leqslant n$, denotes any translate of an m-dimensional linear subspace of E_n. Let S be contained in the m-flat V but in no $(m-1)$-flat. V is uniquely determined by S. m is called the dimension of S, and we write $\dim S = m$.

THEOREM 3. *Let* $\tilde{S}$ *be y-convex and* $\dim S = n$. *If the union of any* $n+2$ *distinct vertical sections of* $\tilde{S}$ *satisfies t.p., then so does* $\tilde{S}$.

Proof. For $n = 1$, this theorem is proved in [4]. The proof goes through for arbitrary n, and we present it here for the sake of completeness.

Analytically, the condition $\dim S = n$ means that $\exists n+1$ points $p^1, \ldots, p^{n+1} \in S \ni$

$$D = \begin{vmatrix} p_1^1 & \cdots & p_n^1 & 1 \\ \cdots & \cdots & \cdots & \\ p_1^{n+1} & \cdots & p_n^{n+1} & 1 \end{vmatrix} \neq 0. \tag{2.2}$$

Suppose hat $|S| = n+1$. Choose $(p^i, y^i) \in \tilde{S}_{p^i}$, $1 \leqslant i \leqslant n+1$. Since $D \neq 0$, the equations

$$p_1^i a_1 + \cdots + p_n^i a_n + a_{n+1} = y^i, \qquad 1 \leqslant i \leqslant n+1, \tag{2.3}$$

have a unique solution $(a_1, \ldots, a_{n+1})$. The $n+1$ vertical sections of $\tilde{S}$ are met by the n.v.h. $y = a_1 x_1 + \cdots + a_n x_n + a_{n+1}$, and so $\tilde{S}$ satisfies t.p.

Suppose next that $|S| \geqslant n+2$. For $p \in S$, let Π_p be the set of n.v.h.'s meeting $\tilde{S}_p = \{\tilde{x}: x = p,\ c_p \leqslant y \leqslant d_p\}$. The n.v.h. $y = a_1 x_1 + \cdots + a_n x_n + a_{n-1}$ may be identified with the point $(a_1,..., a_{n+1}) \in E_{n+1}$. In this identification,

$$\Pi_p = \{(a_1,..., a_{n+1}): c_p \leqslant p_1 a_1 + \cdots + p_n a_n + a_{n+1} \leqslant d_p\}, \tag{2.4}$$

which is a closed convex subset of E_{n+1}.

We conclude from (2.2), (2.4) that $\bigcap_{i=1}^{n+1} \Pi_{p^i}$ is a nonempty bounded subset of E_{n+1}. By assumption, any $n+2$ of the Π_p's have nonempty intersection. Theorem 3 follows from Helly's Theorem.

THEOREM 4. *Let $\tilde{S}$ be y-convex. Suppose that t.p. holds for the union of any $k+2$ distinct vertical sections $\tilde{S}_{p^1},..., \tilde{S}_{p^{k+2}}$ with* $\dim\{{}^1,..., p^{k+2}\} = k$, k *varying from* 1 *to n. Then $\tilde{S}$ satisfies t.p.*

Proof. For $n = 1$, Theorem 4 is identical in meaning with Theorem 3. Assume then that $n \geqslant 2$ and Theorem 4 is true in dimension $<n$. If $\dim S = m < n$, then S spans an m-flat V and $\tilde{S} \subset \tilde{V} = \{\tilde{x}: x \in V\}$. The n.v.h.'s in $\tilde{V}$ are the n.v.h.'s in E_{n+1} intersected with $\tilde{V}$. It follows from the induction hypothesis that $\tilde{S}$ satisfies t.p.

Suppose next that $\dim S = n$. Let $\tilde{S}_{p^1},..., \tilde{S}_{p^{n+2}}$ be distinct vertical sections of $\tilde{S}$. $\bigcup_{i=1}^{n+2} \tilde{S}_{p^i}$ satisfies the conditions of Theorem 4. If $\dim\{p^1,..., p^{n+2}\} < n$, then by the induction hypothesis, $\bigcup_{i=1}^{n+2} \tilde{S}_{p^i}$ satisfies t.p. If $\dim\{p^1,..., p^{n+2}\} = n$, then $\bigcup_{i=1}^{n+2} \tilde{S}_{p^i}$ is assumed to satisfy t.p. It follows from Theorem 3 that $\tilde{S}$ satisfies t.p.

THEOREM 5. *$\tilde{S} \in H_1$ iff $\mathscr{h}_y(\tilde{S})$ satisfies t.p.*

Proof. Let $\tilde{S} \in H_1$. We show that $\mathscr{h}_y(\tilde{S})$ satisfies the conditions of Theorem 4 and so t.p. holds for $\mathscr{h}_y(\tilde{S})$. Let $p^1,..., p^{k+2}$ and $\dim\{p^1,..., p^{k+2}\} = k$, $1 \leqslant k \leqslant n$. We must show that t.p. holds for $\bigcup_{i=1}^{k+2} \mathscr{h}(\tilde{S}_{p^i})$. $p^1,..., p^{k+2}$ lie in some k-flat. After a possible relabeling of indices, these points may be separated into two nonempty sets $\{p^1,..., p^r\}$, $\{p^{r+1},..., p^{k+2}\}$, $1 \leqslant r < k+2$, $\exists \mathscr{h}(p^1,..., p^r) \cap \mathscr{h}(p^{r+1},..., p^{k+2}) \neq \varnothing$ [2, p. 34]. Let $\varepsilon(p^i) = +1$, $1 \leqslant i \leqslant r$, and $\varepsilon(p^i) = -1$, $r+1 \leqslant i \leqslant k+2$. By Remark (i) ε is an inadmissible signature of degree 1 in S. It follows from Theorem 2 that $\tilde{\varepsilon}$ is an inadmissible lifted signature of degree 1 in $\tilde{S}$. Again by Remark (i),

$$\mathscr{h}(\tilde{S}_{p^1},..., \tilde{S}_{p^r}) \cap \mathscr{h}(\tilde{S}_{p^{r+1}},..., \tilde{S}_{p^{k+2}}) \neq \varnothing.$$

We have

$$\mathscr{h}(\tilde{S}_{p^1},..., \tilde{S}_{p^r}) = \bigcup \mathscr{h}(\tilde{p}^1,..., \tilde{p}^r), \tag{2.5}$$

$\tilde{p}_i$ varying over $\mathscr{h}(\tilde{S}_{pi})$, $1 \leqslant i \leqslant r$, with a similar formula for $\mathscr{h}(\tilde{S}_{p^{r+1}},\ldots,\tilde{S}_{p^{k+2}})$. Hence,

$$\sum_{i=1}^{r} \lambda_i \tilde{p}^i = \sum_{i=r+1}^{k+2} \lambda_i \tilde{p}^i \tag{2.6}$$

for some choice of $\tilde{p}^i = (p^i, y^i) \in \mathscr{h}(\tilde{S}_{pi})$ and $\lambda_i \geqslant 0$, $\sum_{i=1}^{r} \lambda_i = \sum_{i=r+1}^{k+2} \lambda_i = 1$, $1 \leqslant i \leqslant k+2$. Subtracting $\tilde{p}^{k+2}$ from both sides of (2.6), we get

$$\sum_{i=1}^{r} \lambda_i (\tilde{p}^i - \tilde{p}^{k+2}) = \sum_{i=r+1}^{k+1} \lambda_i (\tilde{p}^i - \tilde{p}^{k+2}). \tag{2.7}$$

At least one of the λ_i's on the left side of (2.7) is >0, so that $\tilde{p}^1 - \tilde{p}^{k+2},\ldots,\tilde{p}^{k+1} - \tilde{p}^{k+2}$ are linearly dependent. $\dim\{p^1,\ldots,p^{k+2}\} = k$ means that there are k linearly independent vectors among $p^1 - p^{k+2},\ldots, p^{k+1} - p^{k+2}$. It follows that there are k linearly independent vectors among $\tilde{p}^1 - \tilde{p}^{k+2},\ldots,\tilde{p}^{k+1} - \tilde{p}^{k+2}$. We conclude from linear algebra that the respective dimensions of the solution spaces of

$$\sum_{j=1}^{n} (p_j^i - p_j^{k+2})\, a_j = 0, \qquad 1 \leqslant i \leqslant k+1, \tag{2.8}$$

and

$$\sum_{j=1}^{n+1} \left(\tilde{p}_j^i - \tilde{p}_j^{k+2}\right) a_j = \sum_{j=1}^{n} \left(p_j^i - p_j^{k+2}\right) a_j + (y^i - y^{k+2})\, a_{n+1} = 0,$$
$$1 \leqslant i \leqslant k+1, \tag{2.9}$$

are $n-k$ and $n+1-k$. If $a_{n+1} = 0$, then (2.9) becomes (2.8). Thus the space satisfying both (2.9) and $a_{n+1} = 0$ has dimension $n-k$. It follows that (2.9) has a solution $(a_1,\ldots,a_{n+1})$, $a_{n+1} \neq 0$. Letting $a_{n+2} = -\sum_{j=1}^{n+1} a_j \tilde{p}_j^{k+2}$, we conclude that $\tilde{p}^1,\ldots,\tilde{p}^{k+2}$ lie in the n.v.h. $\sum_{k=1}^{n} a_k x_k + a_{n+1} y + a_{n+2} = 0 = 0$. That is, $\bigcup_{i=1}^{k+2} \mathscr{h}(\tilde{S}_{pi})$ satisfies t.p.

Conversely, let t.p. hold for $\mathscr{h}_y(\tilde{S})$. Let $y = l(x)$, l linear, be an n.v.h. meeting all vertical sections of $\mathscr{h}_y(\tilde{S})$. Let $\varepsilon(x)$ be a signature on S with carrier Σ, and $\tilde{\varepsilon}(\tilde{x})$ its lifting to $\tilde{S}$. Suppose $\tilde{\varepsilon}$ is admissible of degree 1. That is, $\exists$ a linear function $L(\tilde{x})$ which is >0 on $\tilde{S}_x$, $x \in \Sigma_+$, and $L(\tilde{x}) < 0$ on $\tilde{S}_x$, $x \in \Sigma_-$. By linearity, we also have $L(\tilde{x}) > 0$ on $\mathscr{h}(\tilde{S}_x)$, $x \in \Sigma_+$, and $L(\tilde{x}) < 0$ on $\mathscr{h}(\tilde{S}_x)$, $x \in \Sigma_-$. $l_1(x) = L(x, l(x))$ is a linear function satisfying $l_1(x) > 0$, $x \in \Sigma_+$, and $l_1(x) < 0$, $x \in \Sigma_-$. That is, ε is admissible of degree 1. Thus, inadmissible signatures of degree 1 in S lift to inadmissible signatures of degree 1 in $\tilde{S}$. We conclude from Theorem 2 that $\tilde{S} \in H_1$.

THEOREM 6. (i) *Let* $1 \leqslant k < \infty$. *If* $\tilde{S} \in H_k$, *then* $\mathscr{h}_y(\tilde{S})$ *satisfies t.p.*

(ii) *If* $\tilde{S}$ *is y-convex and satisfies t.p., then* $\tilde{S} \in H$.

Proof. Since $H_k \subset H_1$, $1 \leqslant k < \infty$, (i) follows from Theorem 5. To prove (ii), we must show $\tilde{S} \in H_k$, $1 \leqslant k < \infty$. We duplicate the argument in the last paragraph of the proof of Theorem 5, replacing the linear function $L(\tilde{x})$ by a polynomial $p_k(\tilde{x})$ of degree k.

3.

In Theorem 6, (ii) is the converse of (i) provided $\tilde{S}$ is assumed y-convex. We give examples showing that the converse to (i) need not hold for general $\tilde{S}$. The sets in Theorem 7 are in E_2 and their y-convex hulls satisfy t.p.

THEOREM 7. *Let* $\tilde{S}^m = \{(x, y) : x^m + y^m = 1\}$, *$m$ a positive even integer.* $\tilde{S}^2 \in H$ *and* $\tilde{S}^m \notin H_2$, $m > 2$.

Proof. We use the criterion of Theorem 2'. Consider first $\tilde{S}^2$. Let $-1 \leqslant x_1 < \cdots < x_{k+2} \leqslant 1$, $1 \leqslant k < \infty$. Suppose $p_k(x, y)$ is a polynomial of degree $\leqslant k$ satisfying $(-1)^i p_k(x, y) > 0$ on $\tilde{S}^2_{x_i}$, $1 \leqslant i \leqslant k+2$. Let $a_i = (\cos\theta_i, \sin\theta_i)$, $b_i = (\cos\theta_i, -\sin\theta_i)$, $0 \leqslant \theta_{k+2} < \cdots < \theta_1 \leqslant \pi$, be, respectively, the upper and lower end points of $\tilde{S}^2_{x_i}$. Then $\operatorname{sgn} p_k(a_i) = \operatorname{sgn}(p_k(b_i) = (-1)^i$. p_k has at least one zero between consecutive a_i's and consecutive b_i's. Hence p_k has at least $k+1$ zeros on each of the semicircles $\tilde{S}_2 \cap \{y > 0\}$, $\tilde{S}_2 \cap \{y < 0\}$, and $q_k(\theta) = p_k(\cos\theta, \sin\theta)$ is a trigonometric polynomial of degree $\leqslant 2k$, having at least $2k+2$ zeros in $(0, 2\pi)$. This contradicts the fact that a trigonometric polynomial of degree $\leqslant k$ has at most $2k$ zeros in $(0, 2\pi)$ [3, p. 77, problem 14]. Hence $p_k(x, y)$ does not exist, and $\tilde{S}^2 \in H_k$, $1 \leqslant k < \infty$, or $\tilde{S}^2 \in H$.

Consider next $m > 2$. Let $p_m(x, y) = -x^2 - y^2 + 2^{1/2 - 1/m}$ and $\{x_1, x_2, x_3, x_4, x_5\} = \{-1, -2^{-1/m}, 0, 2^{-1/m}, 1\}$. We have $(-1)^i p_m(x, y) > 0$ on $\tilde{S}^m_{xi}$, $1 \leqslant i \leqslant 5$, so that $\tilde{S}^m \notin H_2$.

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